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# TWO-DIMENSIONAL THEORIES DEDUCED FROM THREE-DIMENSIONAL THEORY FOR A TRANSVERSELY ISOTROPIC BODY—I. PLATE PROBLEMS

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**Abstract**—The problem of deducing two-dimensional theories from the three-dimensional theory for a transversely isotropic body is investigated. It is shown that the spatial displacements of a three-dimensional body can be represented by the mid-plane displacements and their derivatives and that the general deformation can be decomposed to two independent parts: the asymmetric deformation (*plate problems*) and the symmetric part (*plane problems*). The exact equations for the homogeneous transversely isotropic plates and the approximate equations for the transversely isotropic plates under transverse loads are derived directly from the three-dimensional theory. Torsion of a rectangular plate and bending of an infinitely large plate with a circular hole are examined to illustrate the application of the plate theory developed.

## 1. INTRODUCTION

Since the publication of the excellent work of Cheng (1977, 1979) on deducing the (static) plate theory directly from the three-dimensional theory of elasticity, several extensions have been found in the static problems of both isotropic plates (Barrett and Ellis, 1988) and transversely isotropic plates (Wang, 1985), as well as in the dynamic problems of isotropic plates (Wang, 1988a) and micropolar plates (Wang, 1988b). The significance of Cheng's method is that it opens a systematic way of developing approximate two-dimensional theories from the fundamental three-dimensional theory for many physical problems.

In his paper (1979), Cheng has shown how a refined theory for the bending of isotropic plates can be deduced from the three-dimensional theory of elasticity without any *ad hoc* assumptions. People may doubt the legitimacy of manipulations performed on differential operators in the derivation; however, the final results obtained by his method can be justified by the satisfaction of all equations in the three-dimensional theory. The only approximation in Cheng's plate theory is due to the approximate specification of boundary conditions at the edges of plates; therefore, regarding Saint-Venant's principle, Cheng's theory is a very accurate one.

A parallel development of Cheng's plate theory has been obtained by Wang (1985) for transversely isotropic plates. Instead of using a redundant general solution to the three-dimensional theory of elasticity, as in Cheng (1979), Wang has used Hu's general solution to the three-dimensional elasticity for a transversely isotropic body in his derivation. The refined theory for the transversely isotropic plates obtained there has the identical structure as that of Cheng's theory for the isotropic plates, i.e. a biharmonic equation, a shear equation, and a transcendental equation. The result also indicates that the effect of the transversely isotropic material property is reflected only in the coefficients associated with the higher order terms of the plate thickness in the biharmonic equation and in the shear equation. It should be noted that the three-dimensional displacements there have not been expressed purely by the mid-plane displacements and their derivatives; instead, an auxiliary function was used. Recently, Wang (1988a, b) has generalized his result to obtain the approximate two-dimensional theories for the dynamical problems of isotropic plates and micropolar plates.

Another new development of Cheng's theory is given in a recent paper by Barrett and Ellis (1988) for the isotropic plates under a transverse load (only homogeneous cases are considered in the previous works). Their work actually indicates that various approximate theories for plates subject to surface loads can be developed directly from the three-dimensional theory of elasticity. The paper also presents a detailed discussion on the specification of boundary conditions in light of the work of Gregory and Wan (1984, 1985). Moreover, the relationship between Cheng's theory and other well-known approximate plate theories, such as those of Kirchhoff, Mindlin, Reissner, and Hencky, is well addressed.

This paper extends the work of Wang (1985) and Barrett and Ellis (1988) for the transversely isotropic plates. Firstly, we express the total displacements in terms of mid-plane displacements and their derivatives for the general deformation, a result which has not been achieved by Wang (1985). Secondly, we show that the general deformation of a three-dimensional body can be decomposed to two independent parts: the asymmetric deformation (*plate problems*) and the symmetric deformation (*plane problems*). The plane problems will be discussed in a companion paper (Wang, 1989a). Thirdly, we present the exact equations for the homogeneous plates and the approximate equations for the plates under transverse loads. Finally, two examples, torsion of a rectangular plate and bending of an infinitely large plate with a circular hole, are examined to illustrate the application of our bending theory of transversely isotropic plates.

## 2. BASIC EQUATIONS FOR TRANSVERSELY ISOTROPIC BODY AND HU'S GENERAL SOLUTION

Let us consider a linear and transversely isotropic elastic body occupied the domain $\Omega \times \{-h/2 \leqslant z \leqslant h/2\}$ in a Cartesian coordinate system $(x, y, z)$, with $\Omega$ as an arbitrary region on $(x, y)$ plane. The basic equations for the transversely isotropic body in the three-dimensional linear elasticity are described to be (Lekhniskii, 1981),

*Constitutive equations:*

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{pmatrix} = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{12} & A_{11} & A_{13}\\ A_{13} & A_{13} & A_{33}\end{bmatrix}\begin{pmatrix}\partial_x U\\ \partial_y V\\ \partial_z W\end{pmatrix} \tag{1}$$

$$\sigma_{zx} = A_{44}(\partial_x W + \partial_z U), \quad \sigma_{zy} = A_{44}(\partial_y W + \partial_z V), \quad \sigma_{xy} = A_{66}(\partial_y U + \partial_x V)$$

where $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$ are normal stresses, $\sigma_{zx}$, $\sigma_{zy}$, $\sigma_{xy}$ are shear stresses, and $U$, $V$, $W$ are displacements in the respective Cartesian directions. $A_{ij}$ are material constants with $A_{66} = (A_{11} - A_{12})/2$. The relation between the material constants $A_{ij}$ and the customary engineering material constants is given in Appendix A. The differential operators are defined as

$$\partial_x(\ ) = \frac{\partial(\ )}{\partial x}, \quad \partial_y(\ ) = \frac{\partial(\ )}{\partial y}, \quad \partial_z(\ ) = \frac{\partial(\ )}{\partial z}.$$

*Equilibrium equations* in terms of displacements:

$$\begin{bmatrix} A_{11}\partial_{xx} + A_{66}\partial_{yy} + A_{44}\partial_{zz} & (A_{12}+A_{66})\partial_{xy} & (A_{13}+A_{44})\partial_{zx}\\ (A_{12}+A_{66})\partial_{xy} & A_{66}\partial_{xx} + A_{11}\partial_{yy} + A_{44}\partial_{zz} & (A_{13}+A_{44})\partial_{zy}\\ (A_{13}+A_{44})\partial_{zx} & (A_{13}+A_{44})\partial_{zy} & A_{44}\partial_{xx} + A_{44}\partial_{yy} + A_{33}\partial_{zz} \end{bmatrix}\begin{pmatrix}U\\ V\\ W\end{pmatrix} = 0. \tag{2}$$

Hu's *general solution*: the general solution of the three-dimensional eqns (2) found by Hu (1953) can be represented by

$$U = \partial_y f + \partial_{xz} F, \quad V = -\partial_x f + \partial_{yz} F, \quad W = -\alpha \partial_{zz} F - \beta \nabla^2 F \tag{3}$$

where $\nabla^2 = \partial_{xx} + \partial_{yy}$, and the displacement functions $f$ and $F$ satisfy the following two uncoupled equations

$$\begin{aligned} (s_0^2 \nabla^2 + \partial_{zz}) f &= 0 \\ (s_1^2 \nabla^2 + \partial_{zz})(s_2^2 \nabla^2 + \partial_{zz}) F &= 0. \end{aligned} \tag{4}$$

The coefficients $\alpha$, $\beta$, $s_0$, $s_1$, $s_2$, and other coefficients used late are given in Appendix A.

For the isotropic elastic body, the completeness of the general solution (3) in the regions which are convex in the $z$-direction (like the domain considerd here) has been proven by Wang (1981).

### 3. FROM THREE-DIMENSIONAL EQUATIONS TO TWO-DIMENSIONAL EQUATIONS

The formal general solutions of eqn (4) can be found easily to be

$$\begin{aligned} f(x, y, z) &= SN_0 f_a(x, y) + CS_0 f_s(x, y) \\ F(x, y, z) &= \sum_{i=1}^{2} [SN_i F_{si}(x, y) + CS_i F_{ai}(x, y)] \end{aligned}$$

where $f_a$, $f_s$, $F_{si}$, $F_{ai}$, $i = 1, 2$, are arbitrary functions of $x$ and $y$, with a subscript "$a$" indicates that the corresponding function causes the asymmetric deformation about the middle plane $z = 0$ and a subscript "$s$" indicates that the corresponding function causes the symmetric deformation. The operators $SN_i$, $CS_i$, $i = 0, 1, 2$, which must be interpreted as representing series in powers of operators $(s_i \nabla)^2$, are given in Appendix B.

The displacements follow from eqn (3)

$$\begin{aligned} U &= \partial_y (SN_0 f_a + CS_0 f_s) + \sum_{i=1}^{2} \partial_x (CS_i F_{si} - s_i^2 \nabla^2 SN_i F_{ai}) \\ V &= -\partial_x (SN_0 f_a + CS_0 f_s) + \sum_{i=1}^{2} \partial_y (CS_i F_{si} - s_i^2 \nabla^2 SN_i F_{ai}) \\ W &= \sum_{i=1}^{2} B_w^i \nabla^2 (SN_i F_{si} + CS_i F_{ai}) \end{aligned} \tag{5}$$

where $B_w^i = \alpha s_i^2 - \beta$.

The mid-plane displacements and their derivatives can be found to be

$$\begin{aligned} u &= U|_{z=0} = \partial_y f_s + \sum_{i=1}^{2} \partial_x F_{si}, & \theta &= \partial_z U|_{z=0} = \partial_y f_a - \sum_{i=1}^{2} \partial_x s_i^2 \nabla^2 F_{ai} \\ v &= V|_{z=0} = -\partial_x f_s + \sum_{i=1}^{2} \partial_y F_{si}, & \phi &= \partial_z V|_{z=0} = -\partial_x f_s - \sum_{i=1}^{2} \partial_y s_i^2 \nabla^2 F_{ai} \\ w &= W|_{z=0} = \sum_{i=1}^{2} B_w^i \nabla^2 F_{ai}, & \psi &= \partial_z W|_{z=0} = \sum_{i=1}^{2} B_w^i \nabla^2 F_{si}. \end{aligned} \tag{6}$$

To express the spatial displacements in terms of the mid-plane displacements and their derivatives, we define

$$\zeta = \sum_{i=1}^{2} F_{si}, \quad \eta = \sum_{i=1}^{2} s_i^2 \nabla^2 F_{ai} \tag{7}$$

then

$$\partial_x f_s = \partial_y \zeta - v, \quad \partial_y f_s = u - \partial_x \zeta; \quad \partial_x f_a = -\partial_y \eta - \phi, \quad \partial_y f_a = \partial_x \eta + \theta; \tag{8}$$

since $\partial_{xy}(\ ) = \partial_{yx}(\ )$, it follows that

$$\partial_x u + \partial_y v - \nabla^2 \zeta = 0; \quad \partial_x \theta + \partial_y \phi + \nabla^2 \eta = 0. \tag{9}$$

From eqns (6) and (7), the following relations can be found

$$\nabla^2 F_{s1} = \frac{1}{\alpha(s_1^2 - s_2^2)}(-B_w^2 \nabla^2 \zeta + \psi), \quad \nabla^2 F_{s2} = \frac{1}{\alpha(s_1^2 - s_2^2)}(B_w^1 \nabla^2 \zeta - \psi) \tag{10}$$

$$\nabla^2 F_{a1} = \frac{1}{\beta(s_1^2 - s_2^2)}(-B_w^2 \eta + s_2^2 w), \quad \nabla^2 F_{a2} = \frac{1}{\beta(s_1^2 - s_2^2)}(B_w^1 \eta - s_1^2 w). \tag{11}$$

Substituting expressions (10) and (11) with the eqn (9) into (5), then after some manipulation, we arrive at the representation of the displacements $U$, $V$, and $W$ in terms of the middle plane displacements $u$, $v$, $w$ and rotations $\theta$, $\phi$, $\psi$ as

$$\begin{aligned}
U &= SN_0\theta + \partial_x L_\theta(\partial_x\theta + \partial_y\phi) + \partial_x L_{1w} w + CS_0 u + \partial_x L_u(\partial_x u + \partial_y v) - \partial_x L_{1\psi}\psi \\
V &= SN_0\phi + \partial_y L_\theta(\partial_x\theta + \partial_y\phi) + \partial_y L_{1w} w + CS_0 v + \partial_y L_u(\partial_x u + \partial_y v) - \partial_y L_{1\psi}\psi \\
W &= L_w w + L_{2\theta}(\partial_x\theta + \partial_y\phi) + L_\psi \psi - L_{2u}(\partial_x u + \partial_y v)
\end{aligned} \tag{12}$$

where the differential operators $L_{ij}$ are

$$\begin{aligned}
L_\theta &= s_0^2 SS_0 - \Omega_{ss}^2 + \frac{\alpha}{\gamma}\Omega_{ss}^1, & L_u &= s_0^2 CC_0 - \frac{\beta}{\alpha}\Omega_{cc}^1 + \frac{\beta}{\gamma}\Omega_{cc}^0 \\
L_w &= 1 - \frac{\nabla^2}{\gamma}(\alpha\Omega_{cc}^1 - \beta\Omega_{cc}^0), & L_\psi &= z - \nabla^2\left(\Omega_{ss}^2 - \frac{\beta}{\alpha}\Omega_{ss}^1\right) \\
L_{1w} &= \frac{\nabla^2}{\gamma}\Omega_{ss}^1, & L_{2\theta} &= \frac{1}{\gamma}[\alpha\gamma\Omega_{cc}^2 - (\alpha^2 + \beta\gamma)\Omega_{cc}^1 + \alpha\beta\Omega_{cc}^0] \\
L_{1\psi} &= \frac{1}{\alpha}\Omega_{cc}^1, & L_{2u} &= \frac{\beta}{\alpha\gamma}\nabla^2[\alpha\gamma\Omega_{ss}^2 - (\alpha^2 + \beta\gamma)\Omega_{ss}^1 + \alpha\beta\Omega_{ss}^0].
\end{aligned} \tag{13}$$

The operators $SS_i$, $CC_i$, $\Omega_{ss}^i$, $\Omega_{cc}^i$ are given in Appendix B.

To verify the correctness of the above expressions, let's consider a special case, i.e. when the transversely isotropic body reduces to an isotropic body. In this case, it can be shown that $s_0 = s_1 = s_2 = 1$ and the differential operators $L_\theta$, $L_{2\theta}$, $L_w$, and $L_{1w}$ in (13) have the forms

$$\begin{aligned}
L_\theta &\to \frac{1}{4(1-\nu)\nabla^2}\left(z \cos \nabla z - \frac{\sin \nabla z}{\nabla}\right), & L_{2\theta} &\to -\frac{z \sin \nabla z}{4(1-\nu)\nabla} \\
L_w &\to \cos \nabla z + \frac{z \sin \nabla z}{4(1-\nu)\nabla}\nabla^2, & L_{1w} &\to -\nabla^2 L_\theta.
\end{aligned}$$

Considering only the asymmetric deformation, we have $f_s = 0$, $F_{si} = 0$, $i = 1, 2$, therefore, $u = v = \psi = 0$ and the expression (12) reduces to the form

$$\begin{pmatrix} U \\ V \end{pmatrix} = \frac{\sin \nabla z}{\nabla} \begin{pmatrix} \theta \\ \phi \end{pmatrix} + \frac{1}{4(1-\nu)\nabla^2} \left( z \cos \nabla z - \frac{\sin \nabla z}{\nabla} \right) \begin{pmatrix} \partial_x e \\ \partial_y e \end{pmatrix}$$

$$W = (\cos \nabla z) w - \frac{z \sin \nabla z}{4(1-\nu)\nabla} e$$

where $e = \partial_x \theta + \partial_y \phi - \nabla^2 w$. This expression is identical to the result obtained by Cheng (1979) in a different way. Note that due to the fact that Hu's general solution is proper,† in deriving the expression (12), we do not need to impose the function restriction used by Cheng (1979) in his derivation.

The boundary conditions on the two surfaces $z = -h/2$ and $h/2$ are stress conditions, that is

$$\begin{aligned} \sigma_{zz}(x, y, h/2) &= p_t(x, y), \quad \sigma_{zz}(x, y, -h/2) = p_b(x, y) \\ \sigma_{xz}(x, y, h/2) &= q_{xt}(x, y), \quad \sigma_{xz}(x, y, -h/2) = q_{xb}(x, y) \\ \sigma_{yz}(x, y, h/2) &= q_{yt}(x, y), \quad \sigma_{yz}(x, y, -h/2) = q_{yb}(x, y). \end{aligned} \tag{14}$$

Note that by considering the different boundary conditions, we may have different types of the plate or plane equations. However, we will not discuss this issue in the paper. From the constitutive eqn (1), the relevant components of stress can be calculated to be:

$$\begin{aligned} A_{13}^{-1}\sigma_{zz} = {} & (SN_0 + \nabla^2 L_\theta + \mu \partial_z L_{2\theta})(\partial_x \theta + \partial_y \phi) + (\nabla^2 L_{1w} + \mu \partial_z L_w) w \\ & + (CS_0 + \nabla^2 L_u - \mu \partial_z L_{2u})(\partial_x u + \partial_y v) - (\nabla^2 L_{1\psi} - \mu \partial_z L_\psi)\psi \end{aligned} \tag{15}$$

$$\begin{aligned} A_{44}^{-1}\sigma_{xz} = {} & CS_0 \theta + (\partial_z L_\theta + L_{2\theta})\partial_x(\partial_x \theta + \partial_y \phi) + (\partial_z L_{1w} + L_w)\partial_x w \\ & - s_0^2 \nabla^2 SN_0 u + (\partial_z L_u - L_{2u})\partial_x(\partial_x u + \partial_y v) - (\partial_z L_{1\psi} - L_\psi)\partial_x \psi \end{aligned} \tag{16}$$

$$\begin{aligned} A_{44}^{-1}\sigma_{yz} = {} & CS_0 \phi + (\partial_z L_\theta + L_{2\theta})\partial_y(\partial_x \theta + \partial_y \phi) + (\partial_z L_{1w} + L_w)\partial_y w \\ & - s_0^2 \nabla^2 SN_0 v + (\partial_z L_u - L_{2u})\partial_y(\partial_x u + \partial_y v) - (\partial_z L_{1\psi} - L_\psi)\partial_y \psi. \end{aligned} \tag{17}$$

Simple manipulation shows that the boundary conditions (14) lead to the following two systems of uncoupled linear differential equations for the mid-plane displacements and their derivatives $(w, \theta, \phi)$ and $(u, v, \psi)$, respectively:

$$\begin{bmatrix} \Delta_{11}\partial_x & \Delta_{11}\partial_y & \Delta_{13} \\ CS_0 + \Delta_{22}\partial_{xx} & \Delta_{22}\partial_{xy} & \Delta_{33}\partial_x \\ \Delta_{22}\partial_{xy} & CS_0 + \Delta_{22}\partial_{yy} & \Delta_{33}\partial_y \end{bmatrix} \begin{pmatrix} \theta \\ \phi \\ w \end{pmatrix} = \begin{pmatrix} A_{13}^{-1} p_a \\ A_{44}^{-1} q_{xa} \\ A_{44}^{-1} q_{ya} \end{pmatrix} \quad z = h/2 \tag{18}$$

and

$$\begin{bmatrix} \Sigma_{11}\partial_x & \Sigma_{11}\partial_y & -\Sigma_{13} \\ -s_0^2 \nabla^2 SN_0 + \Sigma_{22}\partial_{xx} & \Sigma_{22}\partial_{xy} & -\Sigma_{33}\partial_x \\ \Sigma_{22}\partial_{xy} & -s_0^2 \nabla^2 SN_0 + \Sigma_{22}\partial_{yy} & -\Sigma_{33}\partial_y \end{bmatrix} \begin{pmatrix} u \\ v \\ \psi \end{pmatrix} = \begin{pmatrix} A_{13}^{-1} p_s \\ A_{44}^{-1} q_{xs} \\ A_{44}^{-1} q_{ys} \end{pmatrix} \quad z = h/2 \tag{19}$$

where

$$\Delta_{11} = SN_0 + \nabla^2 L_\theta + \mu \partial_z L_{2\theta}, \quad \Delta_{13} = \nabla^2 L_{1w} + \mu \partial_z L_w, \quad \Delta_{22} = \partial_z L_\theta + L_{2\theta},$$
$$\Delta_{33} = \partial_z L_{1w} + L_w, \quad z = h/2 \tag{20}$$

† A general solution is *proper* if the total order of the differential equations in the general solution is same as that of the original differential equation system.

$$\Sigma_{11} = CS_0 + \nabla^2 L_u - \mu \partial_z L_{2u}, \quad \Sigma_{13} = \nabla^2 L_{1\psi} - \mu \partial_z L_\psi, \quad \Sigma_{22} = \partial_z L_u - L_{2u},$$
$$\Sigma_{33} = \partial_z L_{1\psi} - L_\psi, \quad z = h/2 \tag{21}$$

and

$$p_a = \frac{p_t - p_b}{2}, \quad q_{xa} = \frac{q_{xt} + q_{xb}}{2}, \quad q_{ya} = \frac{q_{yt} + q_{yb}}{2} \tag{22}$$

$$p_s = \frac{p_t + p_b}{2}, \quad q_{xs} = \frac{q_{xt} - q_{xb}}{2}, \quad q_{ys} = \frac{q_{yt} - q_{yb}}{2}. \tag{23}$$

Equations (18) and (19) indicate that the total deformation of the elastic body considered here can be decomposed into two independent parts: the asymmetric deformation caused by the asymmetric surface loads ($p_a, q_{xa}, q_{ya}$); and the symmetric deformation caused by the symmetric surface loads ($p_s, q_{xs}, q_{ys}$). Such kinds of decomposition of deformation are also observed in the vibration and stability analysis of the three-dimensional plates, Wang and He (1985, 1986), Wang (1986), as well as in the vibration of the micropolar plates, Wang (1989b). Clearly, eqn (18) is for the so-called plate problems, and the eqn (19) is for the so-called plane problems (for most of plane problems, $p_s = q_{xs} = q_{ys} = 0$) in elasticity (Timoshenko and Goodier, 1969). From now on, we will concentrate on the plate problem by setting $p_s = q_{xs} = q_{ys} = 0$, therefore $u = v = \psi = 0$ can be assumed. The discussion of the plane problem is given in the companion paper, Wang (1989a). However, it is interesting to note at this point that the customary surface load condition (i.e. a plate is only subject to the normal load $p_t$ at the top face $z = h/2$) considered in the theory of plate will cause the symmetric deformation of the plate since $p_s = p_t/2 \neq 0$ for this case.

Let $D_P$ be the determinant and $D_{ij}$ be the cofactors of the differential operator matrix in (18). As discussed in Cheng (1979), the general solution of eqn (18) can be expressed as

$$\theta = \sum_{i=1}^{3} D_{i1}\Phi_i, \quad \phi = \sum_{i=1}^{3} D_{i2}\Phi_i, \quad w = \sum_{i=1}^{3} D_{i3}\Phi_i \tag{24}$$

$\Phi_i$'s satisfy the differential equations

$$D_P \Phi_i = X_i, \quad i = 1, 2, 3 \tag{25}$$

in which $X_1 = A_{13}^{-1} p_a$, $X_2 = A_{44}^{-1} q_{xa}$, $X_3 = A_{44}^{-1} q_{ya}$. The determinant $D_P$, after some tedious manipulation, is found to be

$$D_P = \nabla^4 C S_0 G_0, \quad z = h/2 \tag{26}$$

where

$$\gamma G_0 = G_{11} + z G_{22} + \nabla^2 G_{33}. \tag{27}$$

$G_{ij}$ are given in Appendix B. For an isotropic elastic body, $G_0$ becomes

$$G_0 = \frac{\alpha - 1 + \mu\beta}{4\gamma} \frac{1}{\nabla^2}\left(h - \frac{\sin \nabla h}{\nabla}\right)$$
$$= \frac{1 - 2v}{4v(1 - v)} \frac{1}{\nabla^2}\left(h - \frac{\sin \nabla h}{\nabla}\right).$$

Therefore, the determinant (26) reduces to the corresponding result by Cheng (1979) in this case. The solutions (24) and (25) will be investigated in the following two sections for the case of free surface loads and the case of normal (transverse) surface load, respectively.

## 4. EXACT PLATE EQUATIONS: NO TRANSVERSE SURFACE LOADS

For the case of the homogeneous boundary conditions ($p_a = q_{xa} = q_{ya} = 0$), the general solutions of the eqn (25) are the sum of the general solutions of the following three governing differential equations:

$$\nabla^4\Phi_i = 0, \quad \cos(s_0\nabla h/2)\Phi_i = 0, \quad G_0\Phi_i = 0$$

we call the first equation as the *biharmonic equation* and the second one as the *shear equation*, according to the corresponding deformations described by them.

### A. *Biharmonic equation and biharmonic solution*

Let $\Phi_2 = \Phi_3 = 0$, eqns (24) and (25) lead to ($\Phi = \Phi_1$)

$$\nabla^4\Phi = 0$$

$$\theta = D_{11}\Phi, \quad \phi = D_{12}\Phi, \quad w = D_{13}\Phi \tag{28}$$

and the cofactor $D_{ij}$'s are given in Appendix B. It can be shown that

$$\begin{pmatrix}\theta\\ \phi\end{pmatrix} = -\left[1-\left(\frac{\alpha-1}{\gamma}+s_0^2\right)\frac{h^2}{8}\nabla^2\right]\begin{pmatrix}\partial_x\Phi\\ \partial_y\Phi\end{pmatrix}$$

$$w = \left[1-\left(\frac{\alpha-1+\gamma\beta}{\alpha\gamma}+s_0^2\right)\frac{h^2}{8}\nabla^2\right]\Phi.$$

However, since we have in this case

$$\Phi = \left[1+\left(\frac{\alpha-1+\gamma\beta}{\alpha\gamma}+s_0^2\right)\frac{h^2}{8}\nabla^2\right]w$$

therefore,

$$\nabla^4 w = 0 \tag{29}$$

$$\begin{pmatrix}\theta\\ \phi\end{pmatrix} = -\left[1+\frac{\alpha_G h^2}{4(1-\nu)}\nabla^2\right]\begin{pmatrix}\partial_x w\\ \partial_y w\end{pmatrix} \tag{30}$$

and, from eqn (12), the total displacements can be found to be

$$W = \left[1+\frac{z^2}{2\mu}\nabla^2\right]w \tag{31}$$

$$\begin{pmatrix}U\\ V\end{pmatrix} = -z\left\{1+\frac{h^2}{12(1-\nu)}\left[3\alpha_G+2(\alpha_E-2\alpha_G)\left(\frac{z}{h}\right)^2\right]\nabla^2\right\}\begin{pmatrix}\partial_x w\\ \partial_y w\end{pmatrix} \tag{32}$$

where $\alpha_G = G/G'$, $\alpha_E = E/E'$, $G$ and $G'$ are shear modulus in the plane of isotropy and perpendicular to it, respectively; similarly, $E$, $E'$ are Young's modulus in the corresponding planes.

The eqns (29)–(32) form the biharmonic plate equations and it can be shown that $\Phi_2$ or $\Phi_3$ in eqns (24) and (25) will lead to the same equations for $w$, $\theta$, and $\phi$. The normal stresses and shear stresses can be found to be

$$\sigma_{xx} = z\frac{E}{1-v^2}\left\{\partial_{xx}w + v\partial_{yy}w + \frac{h^2}{2}\left[\frac{\alpha_G}{2} - \frac{2\alpha_G - \alpha_E v'}{3}\left(\frac{z}{h}\right)^2\right]\partial_{xx}\nabla^2 w\right\}$$

$$\sigma_{yy} = z\frac{E}{1-v^2}\left\{v\partial_{xx}w + \partial_{yy}w + \frac{h^2}{2}\left[\frac{\alpha_G}{2} - \frac{2\alpha_G - \alpha_E v'}{3}\left(\frac{z}{h}\right)^2\right]\partial_{yy}\nabla^2 w\right\}$$

$$\sigma_{xy} = -z\frac{E}{1-v^2}\partial_{xy}\left\{1 - v + \frac{h^2}{2}\left[\frac{\alpha_G}{2} - \frac{2\alpha_G - \alpha_E v'}{3}\left(\frac{z}{h}\right)^2\right]\nabla^2\right\}w$$

$$\sigma_{zz} = 0, \quad \sigma_{zx} = -\frac{E}{2(1-v^2)}\left(\frac{h^2}{4} - z^2\right)\partial_x\nabla^2 w, \quad \sigma_{zy} = -\frac{E}{2(1-v^2)}\left(\frac{h^2}{4} - z^2\right)\partial_y\nabla^2 w, \tag{33}$$

and the moments and shears to be

$$M_{xx} = -D\left[\partial_{xx}w + v\partial_{yy}w + \frac{8\alpha_G + \alpha_E v'}{40}h^2\partial_{xx}\nabla^2 w\right]$$

$$M_{yy} = -D\left[v\partial_{xx}w + \partial_{yy}w + \frac{8\alpha_G + \alpha_E v'}{40}h^2\partial_{xx}\nabla^2 w\right]$$

$$M_{xy} = -D\partial_{xy}\left[1 - v + \frac{8\alpha_G + \alpha_E v'}{40}h^2\nabla^2\right]w$$

$$V_x = -D\partial_x\nabla^2 w, \quad V_y = -D\partial_y\nabla^2 w \tag{34}$$

where $D = Eh^3/12(1-v^2)$ and $v$ and $v'$ are the Poisson's ratios in the plane of isotropy and perpendicular to it, respectively.

For the isotropic body, $\alpha_G = \alpha_E = 1$, therefore, all results described above reduce to the results in Cheng (1979). Above results also indicate clearly that the transversely isotropic property of material is reflected only in the higher order terms (i.e. terms with $(z/h)^2$), that is, the terms due to the effect of transverse shearing forces on bending (Timoshenko and Woinowsky-Krieger 1959). Hence the plate equations for the bending of the isotropic plate and the transversely isotropic plate are identical if the higher order terms are neglected, as in the classical plate theory. By the same arguments made in Cheng (1979), the eqns (29)–(34) constitute the *first-order theory of transversely isotropic plates*, which can satisfy two edge conditions along the boundary of the region $\Omega$.

B. *Shear equation and shear solution*

Let $\Phi_1 = \Phi_3 = 0$, eqns (24) and (25) lead to ($\Phi = \Phi_2$)

$$CS_0\Phi = 0$$

$$\theta = D_{21}\Phi, \quad \phi = D_{22}\Phi, \quad w = D_{23}\Phi$$

the cofactor $D_{ij}$'s are given in Appendix B. The equation for $\Phi$ can be satisfied if

$$\left[\nabla^2 - \left(\frac{n\pi}{s_0 h}\right)^2\right]\Phi_n = 0, \quad n = 1, 3, 5, \ldots.$$

Let function $Q_n$ be

$$Q_n = -\frac{h}{n\pi}(\Delta_{11}\Delta_{33} - \Delta_{13}\Delta_{22})\partial_y\Phi_n$$

then

$$\left[\nabla^2-\left(\frac{n\pi}{s_0h}\right)^2\right]Q_n=0,\quad n=1,3,5,\ldots \tag{35a}$$

Therefore, the deformation corresponding to $Q_n$ is

$$\theta_n=\frac{n\pi}{h}\partial_yQ_n,\quad \phi_n=-\frac{n\pi}{h}\partial_xQ_n,\quad w=0 \tag{36a}$$

and

$$U_n=\sin\left(\frac{n\pi z}{h}\right)\partial_yQ_n,\quad V_n=-\sin\left(\frac{n\pi z}{h}\right)\partial_xQ_n,\quad W=0. \tag{37a}$$

The normal stresses and shear stresses can be found as

$$\sigma_{xxn}=-\sigma_{yyn}=2G\sin\left(\frac{n\pi z}{h}\right)\partial_{xy}Q_n,\quad \sigma_{zzn}=0,\quad \sigma_{xyn}=G\sin\left(\frac{n\pi z}{h}\right)(\partial_{yy}Q_n-\partial_{xx}Q_n)$$

$$\sigma_{zxn}=G'\frac{n\pi}{h}\cos\left(\frac{n\pi z}{h}\right)\partial_yQ_n,\quad \sigma_{zyn}=-G'\frac{n\pi}{h}\cos\left(\frac{n\pi z}{h}\right)\partial_xQ_n. \tag{38a}$$

The moments and shears are

$$M_{xx}=-M_{yy}=(-1)^{n-1/2}4\left(\frac{h}{n\pi}\right)^2G\partial_{xy}Q_n,\quad M_{xy}=(-1)^{n-1/2}2\left(\frac{h}{n\pi}\right)^2G(\partial_{yy}Q_n-\partial_{xx}Q_n)$$

$$V_{xn}=2(-1)^{n-1/2}G'\partial_yQ_n,\quad V_{yn}=-2(-1)^{n-1/2}G'\partial_xQ_n. \tag{39a}$$

Having obtained these results, one may readily understand the physical meaning of the shear solution and its role in the theory of plates. As for the isotropic plate (Cheng 1979), in the *shear deformation* described by (37a), the middle plane of the plate is subject to no deformation ($U=V=W=0$), planes parallel to the middle plane slide in the same plane and adjacent layers of the plate slide with respect to each other. Hence parallel planes remain parallel and planes at equal distances above and below the midplane of the plate slide with the same displacements but in opposite directions.

For each $n=1,3,\ldots$, the solution (35a) can be considered as an individual term in the Fourier series expansion of the shear deformation. To construct a refined bending theory for the transversely isotropic plate which can satisfy three boundary conditions at each edge of the plate, we need only to consider the leading term $n=1$ in eqns (35a)–(39a). Hence, let $n=1$ and replace $Q_1$ by $2GQ$, eqns (35a)–(39a) now read

$$\left[\nabla^2-\left(\frac{\pi}{s_0h}\right)^2\right]Q=0 \tag{35b}$$

$$\theta=\frac{1+v}{E}\frac{\pi}{h}\partial_yQ,\quad \phi=-\frac{1+v}{E}\frac{\pi}{h}\partial_xQ,\quad w=0 \tag{36b}$$

$$U=\frac{1+v}{E}\sin\left(\frac{\pi z}{h}\right)\partial_yQ,\quad V=-\frac{1+v}{E}\sin\left(\frac{\pi z}{h}\right)\partial_xQ,\quad W=0 \tag{37b}$$

$$\sigma_{xx}=-\sigma_{yy}=\sin\left(\frac{\pi z}{h}\right)\partial_{xy}Q,\quad \sigma_{zz}=0,\quad \sigma_{xy}=\tfrac{1}{2}\sin\left(\frac{\pi z}{h}\right)(\partial_{yy}Q-\partial_{xx}Q)$$

$$\sigma_{zx}=\frac{1}{2\alpha_G}\frac{\pi}{h}\cos\left(\frac{\pi z}{h}\right)\partial_yQ,\quad \sigma_{zy}=-\frac{1}{2\alpha_G}\frac{\pi}{h}\cos\left(\frac{\pi z}{h}\right)\partial_xQ \tag{38b}$$

$$M_{xx} = -M_{yy} = 2\left(\frac{h}{\pi}\right)^2 \partial_{xy} Q, \quad M_{xy} = \left(\frac{h}{\pi}\right)^2 (\partial_{yy} Q - \partial_{xx} Q)$$

$$V_x = \frac{1}{\alpha_G} \partial_y Q, \qquad V_y = -\frac{1}{\alpha_G} \partial_x Q. \tag{39b}$$

Combining the biharmonic solution (29)–(34) and the shear solution (35b)–(39b), we arrive at a *second-order refined theory* for the bending of the transversely isotropic plates with the two differential governing eqns (29) and (35b). Since the total order of the governing equations is 6, so three boundary conditions at each edge of the plates can be prescribed in the refined plate theory. It is important to note that the three-dimensional equilibrium eqn (2) is satisfied by any solution of the refined plate theory, and the only approximation in the theory is introduced by the approximate specification of the boundary conditions at the edges of the plate (i.e. the boundary conditions are specified in terms of the stress resultants or some combination of midplane displacements and their derivatives, instead of the stress or displacement distribution over the thickness $-h/2 \leqslant z \leqslant h/2$). Therefore, in the cases where Saint Venant's principle holds, the refined theory should be a very accurate one.

## 5. APPROXIMATE PLATE EQUATIONS: TRANSVERSE SURFACE LOADS

Now let us consider the case that the plate is only subject to the transverse surface load, i.e. $q_{xa} = q_{ya} = 0$. Since $X_2 = X_3 = 0$, we can set $\Phi_2 = \Phi_3 = 0$, and reduce eqns (25) and (24) to

$$D_P w = A_{13}^{-1} D_{13} p_s \tag{40}$$

$$D_{13}\theta = D_{11} w, \quad D_{13}\phi = D_{12} w. \tag{41}$$

Equation (40) is the exact governing equation for the normal displacement $w$ at the midplane of the plate subject to the transverse surface load. Since this equation is of infinite order, however, it is not applicable in most cases. In the following we will try to develop an approximate transversely isotropic plate theory which has the same structure as that of the first-order theory given in the previous section. To this end we need to calculate the series expansions of the differential operators in (40) and (41) up to the fourth-order of the plate thickness $h$. After tedious manipulation, the results turn out to be

$$D_P = \frac{\alpha - 1 + \mu\beta}{24\gamma} h^3 \nabla^4 \left[ 1 - (\alpha_H + \alpha_G) \frac{h^2}{8} \nabla^2 + O(h^4) \right] \tag{42}$$

$$(D_{11}, D_{12}) = -\left[ 1 - \left( \frac{\alpha - 1}{\gamma} + \alpha_G \right) \frac{h^2}{8} \nabla^2 + O(h^4) \right] (\partial_x, \partial_y),$$

$$D_{13} = 1 - \left( \frac{\alpha - 1 + \gamma\beta}{\alpha\gamma} + \alpha_G \right) \frac{h^2}{8} \nabla^2 + O(h^4) \tag{43}$$

where†

$$\alpha_H = \frac{\mu\beta - \alpha - 1}{5\alpha\mu}.$$

Furthermore,

† The calculation for obtaining $\alpha_H$ is performed by the symbolic computation software *Maple*.

$$D_{13}^{-1} = 1+\left(\frac{\alpha-1+\gamma\beta}{\alpha\gamma}+\alpha_G\right)\frac{h^2}{8}\nabla^2+O(h^4)$$

and

$$\left[1-(\alpha_H+\alpha_G)\frac{h^2}{8}\nabla^2+O(h^4)\right]^{-1} = 1+(\alpha_H+\alpha_G)\frac{h^2}{8}\nabla^2+O(h^4).$$

Therefore, by dropping all the terms associated with $h^4$ or the higher orders and considering $A_{13}h^3$ as a constant, we finally arrive at the following equations

$$\nabla^4 w = [1-\alpha_P h^2\nabla^2]\frac{p}{D'} \tag{44}$$

$$\begin{pmatrix}\theta\\ \phi\end{pmatrix} = -\left[1+\frac{\alpha_G h^2}{4(1-\nu)}\nabla^2\right]\begin{pmatrix}\partial_x w\\ \partial_y w\end{pmatrix} \tag{45}$$

where $p = 2p_s = p_t - p_b$ is the customary transverse load used in the plate theory, and

$$\alpha_P = \frac{1}{8}\left(\frac{\alpha-1+\gamma\beta}{\alpha\gamma}-\alpha_H\right),\quad D' = \frac{\alpha-1+\mu\beta}{12\gamma}A_{13}h^3.$$

Equations (44) and (45) form the basic equations for an approximate first-order theory for the bending of the transversely isotropic plates. Similar equation for the mid-plane displacement $w$ has been obtained by Barrett and Ellis (1988) for the isotropic elastic plates. Actually, in this case, we have $D' = D$, $\alpha_H = \frac{2}{5}$, and $\alpha_P = (8-3\nu)/40(1-\nu)$, the eqn (44) reduces to the corresponding result in Barrett and Ellis (1988).

From the eqns (44) and (45), it can be shown that all the expressions about the spatial (or total) displacements, stresses, and stress resultants in the previous section [(31)–(34)] are still valid except the expression for the normal stress $\sigma_{zz}$, which becomes

$$\sigma_{zz} = \frac{p}{2}\left(\frac{z}{h}\right)\left[4-3\left(\frac{z}{h}\right)^2\right] \tag{46}$$

in this case. Clearly, even if people doubt the legitimacy of the manipulation performed on differential operators, the plate equations obtained above can be justified by comparing their forms with the forms of the corresponding equations in other well-known plate theories.

Since the shear solution (35b)–(39b) is still valid after the surface load condition is satisfied approximately by the eqns (44)–(45), the combination of the eqns (44)–(45) and shear solution therefore form a refined theory for the bending of the transversely isotropic plates under the transverse loads, in which three boundary conditions at each edge of the plates can be prescribed. As in Barrett and Ellis (1988), by adopting the works of Gregory and Wan (1984, 1985) into the case of transversely isotropic body, the similar discussion about the specification of the boundary conditions on the edges of the plate can be made. However, this issue will not be addressed here.

## 6. EXAMPLES

To illustrate the applications of the theories developed in the previous sections and compare the results with the known exact and approximate solutions, we present the following two examples. Note that the same examples for the isotropic body have been discussed by Cheng (1979).

A. *Torsion of a rectangular plate*

Consider a rectangular plate with its four faces $y = \mp b$ and $z = \mp h/2$ free of stress while the two ends $x = 0$ and $x = a$ twisted by two equal and opposite couples. From eqn (37a) and the character of the shear solution described in Section 4, we note that the shear solution is related to, and important in, the torsion problem. However, only the shear solution will let the boundary conditions $\sigma_{xy} = 0$ unsatisfied on $y = \mp b$. In order to satisfy this condition, a simple biharmonic solution is introduced. The final torsion solution can be found to be

$$w = cxy, \quad Q = \sum_{n=0}^{\infty} A_n \cosh\left(\frac{\lambda_n y}{s_0}\right), \quad \lambda_n = \frac{(2n+1)\pi}{h} \tag{47}$$

with

$$A_n = \frac{8c(-1)^n s_0^4}{h\lambda_n^4 \cos\left(\frac{\lambda_n b}{s_0}\right)}$$

and the constant $c$ can be determined by couples applied at the ends $x = 0$ and $x = a$. The non-zero displacement and stresses are specified by

$$U = \sum_{n=0}^{\infty} \lambda_n \frac{8c(-1)^n s_0^3}{h\lambda_n^4 \cos\left(\frac{\lambda_n y}{s_0}\right)} \sin(\lambda_n z) \sinh\left(\frac{\lambda_n y}{s_0}\right)$$

$$\sigma_{xy} = -2Gc\left[z - \frac{4s_0^2}{h}\sum_{n=0}^{\infty} \frac{(-1)^n \cosh\left(\frac{\lambda_n y}{s_0}\right)}{\cosh\left(\frac{\lambda_n b}{s_0}\right)} \sin(\lambda_n z)\right]$$

$$\sigma_{xz} = \frac{8Gcs_0}{h}\sum_{n=0}^{\infty} \frac{(-1)^n \sinh\left(\frac{\lambda_n y}{s_0}\right)}{\lambda_n^2 \cosh\left(\frac{\lambda_n b}{s_0}\right)} \cos(\lambda_n z). \tag{48}$$

For the isotropic plate $s_0 = 1$, the solution is identical to the Saint-Venant's exact solution (Fung, 1965). This solution also indicates that basically the part of the torsion solution for the transversely isotropic plates caused by the shear solution can be obtained from the corresponding part for the isotropic plates by replacing the width $b$ with an "equivalent width" $b/s_0$; and that the part of the solution corresponding to the result of classical plate theory is same for both isotropic and transversely isotropic plates, since the transversely isotropic property has no effect in the classical plate theory as having been pointed out previously.

B. *The effect of holes on stress distributions in plates*

Consider an infinitely large plate with a circular hole of radius $r_0$. The boundary conditions to be fulfilled are: $x = \pm\infty$, $M_{xx} = M_0$, $M_{xy} = 0$, $V_x = 0$; $y = \pm\infty$, $M_{yy} = 0$, $M_{xy} = 0$, $V_y = 0$; and $r = r_0$, $M_{rr} = 0$, $M_{r\theta} = 0$, $V_r = 0$. This problem has been investigated by several researchers using the theory of elasticity and the improved plate theories for the isotropic plates, Alblas (1957), Lee and Conlee (1968), Reissner (1975), and Cheng (1979), and for the transversely isotropic plates, Wang (1985). Here we just present the major result in Wang (1985).

Firstly the boundary conditions at infinity are transformed to the forms in the polar coordinate:

$$r=\infty,\quad M_{rr}=M_0\frac{1+\cos 2\theta}{2},\quad M_{r\theta}=-M_0\frac{\sin 2\theta}{2},\quad V_r=0.$$

The shear and biharmonic solutions selected are

$$\frac{Q}{M_0}=cK_2(\xi r)\sin 2\theta=cK_2(\rho)\sin 2\theta,\quad \frac{Dw}{M_0}=c_1r^2+c_2\ln r+(c_3r^2+c_4r^{-2}+c_5)\cos 2\theta \quad (49)$$

where $\xi=\pi/\sqrt{\alpha_G}h$, $\rho=\xi r$, and $K_i$, $i=0,1,2$, is the modified Bessel function of the second kind of order $i$. After applying the prescribed boundary conditions at $r=r_0$ and $r=\infty$, we obtain

$$c=-\frac{2\alpha_G}{H},\quad c_1=-\frac{1}{4(1+\nu)},\quad c_2=-\frac{r_0^2}{2(1-\nu)},\quad c_3=-\frac{1}{4(1-\nu)},\quad c_5=-\frac{r_0^2K_2}{2H}$$

$$c_4=\frac{1}{12(1-\nu)}\frac{r_0^4}{H}\left[2K_0+\frac{16}{\rho_0}\left(K_1+\frac{3K_2}{\rho_0}\right)+(1-3\nu)K_2-\tfrac{3}{5}(8\alpha_G+\alpha_E\nu')\left(\frac{h}{r_0}\right)^2K_2\right]$$

where $\rho_0=\xi r_0$, $H=(1+\nu)K_2+2K_0$, and $K_i$, $i=0,1,2$, is the value of $K_i$ at $\rho=\rho_0$. The moment and the stress related with the calculation of concentration factors are the following

$$\frac{M_{\theta\theta}}{M_0}=\frac{1-\cos 2\theta}{2}+\frac{1}{2}\left(\frac{\rho_0}{\rho}\right)^2+\frac{4}{r^2}\left[\frac{3(1-\nu)}{2r^2}c_4+c_5\right.$$
$$\left.-\frac{3}{20}(8\alpha_G+\alpha_E\nu')\left(\frac{h}{r}\right)^2c_5\right]\cos 2\theta-\frac{8}{H}\left[K_1(\rho)+\frac{3K_2(\rho)}{\rho}\right]\frac{\cos 2\theta}{\rho}$$

$$\frac{\sigma_{\theta\theta}(\rho_0,\theta,h/2)}{\sigma_0}=\frac{1-\cos 2\theta}{2}+\frac{1}{2}\left(\frac{\rho_0}{\rho}\right)^2+\frac{4}{r^2}\left[\frac{3(1-\nu)}{2r^2}c_4+c_5-\frac{3}{20}(8\alpha_G+\alpha_E\nu')\left(\frac{h}{r}\right)^2c_5\right]\cos 2\theta$$
$$-\frac{2\pi^2}{3H}\left[K_1(\rho)+\frac{3K_2(\rho)}{\rho}\right]\frac{\cos 2\theta}{\rho} \quad (50)$$

where $\sigma_0=6M_0/h^2$. The *stress couple concentration factor* $k_M$ and the *maximum-stress concentration factor* $k$ are defined as

$$k_M=\frac{M_{\theta\theta}(\rho_0,\pi/2)}{M_0}=1+\frac{2(1+\nu)K_2}{H}=1+\frac{2(1+\nu)K_2(\rho_0)}{(1+\nu)K_2(\rho_0)+2K_0(\rho_0)} \quad (51)$$

$$k_\sigma=\frac{\sigma_{\theta\theta}(\rho_0,\pi/2,h/2)}{\sigma_0}=k_M+\frac{K_2(\rho_0)}{5H}(2\alpha_G-\alpha_E\nu')\left(\frac{h}{r_0}\right)^2$$
$$-\frac{8}{\rho_0H}\left(1-\frac{\pi^2}{12}\right)\left[K_1(\rho_0)+\frac{3K_2(\rho_0)}{\rho_0}\right] \quad (52)$$

respectively. For the isotropic plates, $\alpha_G=\alpha_E=1$ and $\nu'=\nu$, once again the results described above reduce to the corresponding results by Cheng (1979). Comparing the expression of the stress couple concentration factor here with that in Cheng (1979), we find that in the calculation of $k_M$ for a transversely isotropic plate of thick $h$ is exactly equivalent to that for an isotropic plate of thickness $\sqrt{\alpha_G}h$. For very thin plates, $h \ll r_0$, $\rho_0\to\infty$, we find that

$$k_M = \frac{5+3\nu}{3+\nu}$$

which is the result obtained by means of classical plate theory. Since $\alpha_G = G/G'$ and $k_M$ is a decreasing function of $\rho_0$, therefore, with respect to an isotropic plate, when $G' > G$, i.e. when the transverse shear stiffness is greater than the shear stiffness in $(x, y)$ plane, then the stress couple concentration is intensified, while when $G' < G$, the stress couple concentration is reduced. This observation agrees with the intuitive physical consideration. For the maximum-stress concentration factor, comparing with the result for the isotropic plates, we find the following additional term purely due to the non-isotropic material property (vanishing for the isotropic plates)

$$\frac{K_2(\rho_0)}{5H}[2(\alpha_G - 1) - (\alpha_E\nu' - \nu)]\left(\frac{h}{r_0}\right)^2.$$

However, since the maximum-stress concentration factor is dependable only for the thin plates ($h \ll r_0$), this term is of the neglectable magnitude (Wang, 1985). Hence, the remarks just made for the stress couple concentration factor are valid for the maximum-stress concentration factor, too. For a further and detailed discussion of the problem readers are referred to Wang (1985).

## 7. CONCLUSION

In the above sections, a refined two-dimensional theory for the transversely isotropic plates has been deduced systematically and directly from the three-dimensional theory without any *ad hoc* assumptions. It is found that, in the refined plate theory, the effect of transversely isotropic property is reflected only in the higher order terms of the plate thickness in the biharmonic solution and in the shear solution. Hence, for the very thin plates, the transversely isotropic bodies and isotropic bodies exhibit the same bending behavior. For homogeneous plates, the refined plate theory is exact in the sense that a solution of the refined plate theory satisfies all the equations in the three-dimensional theory. For the plates under a transverse load, this fact is no longer held. However, the refined plate theory for the loaded plates can still be justified by comparing its form with that of other well-known modified plate theories for isotropic plates. Furthermore, the two examples studied also indicate that in some cases a solution of the refined theory for a transversely isotropic plate can be obtained approximately from the corresponding solution of the refined theory for an isotropic plate by replacing certain length terms with the "equivalent" terms.

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## APPENDIX A: MATERIAL CONSTANTS

1. *Coefficients used in Hu's general solution*

$$\alpha = \frac{A_{44}}{A_{13}+A_{44}}, \quad \beta = \frac{A_{11}}{A_{13}+A_{44}}, \quad \gamma = \frac{A_{33}}{A_{13}+A_{44}}, \quad \mu = \frac{A_{33}}{A_{13}}$$

$$s_0^2 = \frac{A_{66}}{A_{44}}, \quad s_{12}^2 = \frac{B \pm \sqrt{B^2 - 4A_{11}A_{33}A_{44}^2}}{2A_{33}A_{44}}, \quad B = A_{11}A_{33} - 2A_{13}A_{44} - A_{13}^2$$

$$s_1^2 + s_2^2 = \frac{\alpha^2 + \beta\gamma - 1}{\alpha\gamma}, \quad s_1^2 s_2^2 = \frac{\beta}{\gamma}, \quad \alpha\mu = \mu - \gamma.$$

2. *$A_{ij}$ and the customary engineering material constants*

Let $E$, $G$, and $v$ be Young's modulus, shear modulus, and Poisson's ratio in the plane of isotropy, respectively, and let $E'$, $G'$, and $v'$ be the transverse Young's modulus, shear modulus, and Poisson's ratio, respectively. Then

$$A_{11} = \frac{E(1-\alpha_E v'^2)}{(1+v)(1-v-2\alpha_E v'^2)}, \quad A_{33} = \frac{E'(1-v)}{(1-v-2\alpha_E v'^2)}, \quad A_{44} = G'$$

$$A_{12} = \frac{E(v+\alpha_E v'^2)}{(1+v)(1-v-2\alpha_E v'^2)}, \quad A_{13} = \frac{v'E}{(1-v-2\alpha_E v'^2)}, \quad A_{66} = G.$$

For isotropic materials:

$$\alpha = 1-2v, \quad \beta = \gamma = 2(1-v), \quad \mu = (1-v)/v.$$

## APPENDIX B: DIFFERENTIAL OPERATORS

1. *Basic differential operators*

$$SN_i = \frac{\sin(s_i \nabla z)}{s_i \nabla}, \quad CS_i = \cos(s_i \nabla z), \quad SS_i = \frac{z - SN_i}{s_i^2 \nabla^2}, \quad CC_i = \frac{1 - CS_i}{s_i^2 \nabla^2}, \quad i = 0, 1, 2$$

$$\Omega_{ss}^i = \frac{s_1^{2i} SS_1 - s_2^{2i} SS_2}{s_1^2 - s_2^2}, \quad \Omega_{cc}^i = \frac{s_1^{2i} CC_1 - s_2^{2i} CC_2}{s_1^2 - s_2^2}, \quad i = 0, 1, 2, 3$$

when $s_0 \to 1$, $s_1 \to 1$, $s_2 \to 1$:

$$\Omega_{ss}^i \to (i-1)\frac{z - \dfrac{\sin \nabla z}{\nabla}}{\nabla^2} - \frac{1}{2}\frac{z\cos \nabla z - \dfrac{\sin \nabla z}{\nabla}}{\nabla^2}, \quad \Omega_{cc}^i \to (i-1)\frac{1-\cos \nabla z}{\nabla^2} + \frac{z}{2}\frac{\sin \nabla z}{\nabla}$$

$$\partial_z SN_i = CS_i, \quad \partial_z CS_i = -s_i^2 \nabla^2 SN_i, \quad \partial_z SS_i = CC_i, \quad \partial_z CC_i = SN_i, \quad i = 0, 1 2$$

$$\partial_z \Omega_{ss}^i = \Omega_{cc}^i, \quad \partial_z \Omega_{cc}^i = z\frac{s_1^{2i} - s_2^{2i}}{s_1^2 - s_2^2} - \nabla^2 \Omega_{ss}^{i+1}, \quad i = 0, 1, 2, 3$$

$$\partial_z L_\theta = s_0^2 CC_0 - \Omega_{cc}^2 + \frac{\alpha}{\gamma}\Omega_{cc}^1, \quad \partial_z L_w = \frac{\nabla^4}{\gamma}(\alpha\Omega_{ss}^2 - \beta\Omega_{ss}^1) - \frac{\alpha}{\gamma} z\nabla^2$$

$$\partial_z L_{1w} = \frac{\nabla^2}{\gamma}\Omega_{cc}^1, \quad \partial_z L_{2\theta} = -\frac{z}{\gamma} - \frac{\nabla^2}{\gamma}[\alpha\gamma\Omega_{ss}^3 - (\alpha^2+\beta\gamma)\Omega_{ss}^2 + \alpha\beta\Omega_{ss}^1].$$

*2. $G_{ij}$ and cofactors $D_{ij}$*

$$G_{11} = (1-\gamma\beta-\alpha)\Omega_{ss}^1 + \mu(1-\gamma\beta-\alpha^2)\Omega_{ss}^2 + \alpha\gamma\mu\Omega_{ss}^3, \quad G_{22} = \alpha\beta\Omega_{cc}^0 + \alpha(1+\mu\beta-\alpha)\Omega_{cc}^1 + \alpha\gamma\Omega_{cc}^2$$

$$G_{33} = \beta(1-\mu\beta)(\Omega_{ss}^2\Omega_{cc}^0 - \Omega_{ss}^1\Omega_{cc}^1) + (1-\mu\beta-\alpha)(\Omega_{ss}^2\Omega_{cc}^1 - \Omega_{ss}^1\Omega_{cc}^2) - \alpha\gamma(\Omega_{ss}^2\Omega_{cc}^2 + \Omega_{ss}^3\Omega_{cc}^1) + \alpha\beta\mu\Omega_{ss}^3\Omega_{cc}^0$$

$$D_{11} = -CS_0\Delta_{33}\partial_x, \quad D_{12} = -CS_0\Delta_{33}\partial_y, \quad D_{13} = CS_0(CS_0 + \Delta_{22}\nabla^2)$$

$$D_{21} = CS_0\Delta_{13} - (\Delta_{11}\Delta_{33} - \Delta_{13}\Delta_{22})\partial_{yy}, \quad D_{22} = (\Delta_{11}\Delta_{33} - \Delta_{13}\Delta_{22})\partial_{xy}, \quad D_{23} = -CS_0\Delta_{11}\partial_x$$

$$D_{31} = (\Delta_{11}\Delta_{33} - \Delta_{13}\Delta_{22})\partial_{xy}, \quad D_{32} = CS_0\Delta_{13} - (\Delta_{11}\Delta_{33} - \Delta_{13}\Delta_{22})\partial_{xx}, \quad D_{33} = -CS_0\Delta_{11}\partial_y.$$